

Insecurity of Quantum Bit Commitment with Secret Parameters

Chi-Yee Cheung*

*Institute of Physics, Academia Sinica
Taipei, Taiwan 11529, Republic of China*

The impossibility proof of unconditionally secure quantum bit commitment is crucially dependent on the assertion that Bob is not allowed to generate probability distributions unknown to Alice. This assertion is actually not meaningful, because Bob can always cheat without being detected. In this paper we prove that, for any concealing protocol involving secret probability distributions, there exists a cheating unitary transformation that is known to Alice. Our result closes a gap in the original impossibility proof.

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I. INTRODUCTION

Bit commitment is an important primitive that can be used to implement other two-party cryptographic protocols [1]. In a bit commitment protocol, Alice commits to Bob a secret bit $b \in \{0, 1\}$ that is to be unveiled at a later time. In order to guarantee that she will not change her mind, Alice sends Bob a piece of evidence that can later on be used to verify her honesty when she unveils.

A bit commitment scheme is secure if (1) Bob cannot extract the value of b before Alice unveils it (concealing), and (2) Alice cannot change the value of b without Bob's knowledge (binding). Furthermore, if the scheme remains secure even if Alice and Bob were endowed with capabilities limited only by the laws of nature, then it is said to be unconditionally secure.

In a typical classical bit commitment scheme, Alice writes the committed bit b on a piece of paper and locks it in a strong safe. She then delivers the safe to Bob but keeps the key. Later she unveils by disclosing the bit value and presenting the key to Bob for verification. However such a scheme is clearly not unconditionally secure because its security depends on, among other things, the assumption that Bob cannot open the safe without the help of Alice. In fact all classical bit commitment schemes are based on some unproven assumptions, so that unconditional security is not possible in classical settings.

By introducing quantum mechanics into the bit commitment game, one hopes to achieve unconditional security which is guaranteed by the laws of nature. In a quantum bit commitment (QBC) protocol, Alice and Bob execute a series of quantum and classical operations, such that at the end of the commitment phase, Bob has in his hand a quantum state characterized by a density matrix $\rho_B^{(b)}$. The idea is that, with additional information from Alice in the unveiling phase, Bob can use $\rho_B^{(b)}$ to check whether Alice is honest.

II. NO-GO THEOREM

It is generally believed that Lo and Chau [2, 3] and Mayers [4, 5] proved in 1997 that unconditionally secure QBC is impossible. The arguments can be summarized as follows. First of all, it is observed that

*Electronic address: cheung@phys.sinica.edu.tw

the whole commitment process, which may involve any number of rounds of quantum and classical exchanges between Alice and Bob, can always be represented by an unitary transformation $\mathcal{U}_{AB}^{(b)}$ on an initial pure state $|\phi_{AB}^{(b)}\rangle$ in the combined Hilbert space $H_A \otimes H_B$ of Alice and Bob. Therefore at the conclusion of the commitment process, the overall state is given by

$$|\Psi_{AB}^{(b)}\rangle = \mathcal{U}_{AB}^{(b)} |\phi_{AB}^{(b)}\rangle. \quad (1)$$

The pure state $|\Psi_{AB}^{(b)}\rangle$ is called a purification of the density matrix $\rho_B^{(b)}$ such that

$$\text{Tr}_A |\Psi_{AB}^{(b)}\rangle \langle \Psi_{AB}^{(b)}| = \rho_B^{(b)}. \quad (2)$$

In this approach, Alice and Bob can leave all undisclosed parameters undetermined at the quantum level. Moreover since the reduced density matrix $\rho_B^{(b)}$ on Bob's side is unchanged, he cannot distinguish whether Alice purifies or not.

In order that the protocol is concealing, the density matrices $\rho_B^{(0)}$ and $\rho_B^{(1)}$ must be either equal,

$$\rho_B^{(0)} = \rho_B^{(1)}, \quad (3)$$

or arbitrarily close to each other,

$$\rho_B^{(0)} \approx \rho_B^{(1)}, \quad (4)$$

corresponding respectively to the perfect concealing and near-perfect concealing cases. The closeness between the two density matrices, $\rho_B^{(1)}$ and $\rho_B^{(0)}$, can be described quantitatively by the fidelity $F(\rho_B^{(1)}, \rho_B^{(0)})$. Let $|\Phi_{AB}^{(b)}\rangle$ be any purification of $\rho_B^{(b)}$ so that

$$\text{Tr}_A |\Phi_{AB}^{(b)}\rangle \langle \Phi_{AB}^{(b)}| = \rho_B^{(b)}. \quad (5)$$

Then, according to Uhlmann's Theorem, the fidelity can be expressed as

$$F(\rho_B^{(1)}, \rho_B^{(0)}) = \max |\langle \Phi_{AB}^{(1)} | \Phi_{AB}^{(0)} \rangle|, \quad (6)$$

where the maximization is over all possible purifications, and $0 \leq F(\rho_B^{(1)}, \rho_B^{(0)}) \leq 1$. Note that

$$F(\rho_B^{(1)}, \rho_B^{(0)}) = 1 \quad (7)$$

if and only if the perfect concealing condition, Eq. (3), holds; in this case Bob can extract absolutely no information about Alice's committed bit b from $\rho_B^{(b)}$. In general we have

$$F(\rho_B^{(1)}, \rho_B^{(0)}) = 1 - \delta, \quad (8)$$

where $\delta \geq 0$. For the near-perfect concealing case, Eq. (4), we have $\delta > 0$ and it can be made arbitrarily small by increasing the security parameter N .

It is well known [8] that for a fixed purification $|\Psi_{AB}^{(1)}\rangle$ of $\rho_B^{(1)}$, there exists a purification $|\Phi_{AB}^{(0)}\rangle$ of $\rho_B^{(0)}$, such that

$$|\langle \Psi_{AB}^{(1)} | \Phi_{AB}^{(0)} \rangle| = 1 - \delta. \quad (9)$$

Furthermore since both $|\Phi_{AB}^{(0)}\rangle$ and $|\Psi_{AB}^{(0)}\rangle$ are purifications of the same reduced density matrix $\rho_B^{(0)}$, they are related by an unitary transformation:

$$|\Phi_{AB}^{(0)}\rangle = U_A |\Psi_{AB}^{(0)}\rangle, \quad (10)$$

where U_A acts on Alice's Hilbert space H_A only [8]. In particular, for the perfect concealing case where $\delta = 0$, it is clear from Eqs. (9, 10) that

$$U_A|\Psi_{AB}^{(0)}\rangle = |\Psi_{AB}^{(1)}\rangle, \quad (11)$$

apart from an unimportant phase factor.

The existence of U_A means that Alice can cheat with the following strategy (called EPR attack). To begin with, she always commits to $b = 0$. Later on, right before she unveils, if she wants to keep her initial commitment, she simply follows the protocol honestly to the end. Otherwise if she wants to switch to $b = 1$, she only needs to apply U_A to her share of the state $|\Psi_{AB}^{(0)}\rangle$, and then proceed as if she had committed to $b = 1$ in the first place. In the perfect concealing case, Alice succeeds with probability one. Otherwise, in the near-perfect case, her success probability approaches unity as $N \rightarrow \infty$ ($\delta \rightarrow 0$). Hence if a protocol is concealing, it cannot be binding at the same time. This is the no-go theorem of unconditionally secure quantum bit commitment [2, 3, 4, 5].

III. SECRET PARAMETERS

It has been pointed out that the above proof only establishes the existence of the cheating transformation U_A , but there is no guarantee that U_A is always known to Alice [6, 7]. The point is, even in the fully purified approach, the overall state $|\Psi_{AB}^{(b)}(\omega)\rangle$ may still depend on some probability distribution ω unknown to Alice. If so, then the cheating transformation $U_A(\omega)$ would in general depend on ω , and Alice would not be able to implement $U_A(\omega)$ without the help of Bob. This is a serious logical gap in the original impossibility proof. To overcome this gap, the proof [2, 3, 4, 5] asserts that Alice knows in detail all the probability distributions generated by Bob in any QBC protocol, hence she knows $U_A(\omega)$.

This assertion is actually not correct. As shown in the Appendix, it is not meaningful to specify a probability distribution to an untrustful party (Bob) in a quantum protocol, because he can always cheat without being detected [9]. So, regardless of whether secret parameters are allowed in QBC protocols or not, they are potentially there and must be taken into account in security analysis. Consequently, whether the no-go theorem remains valid in the presence of secret parameters is a crucial question that cannot be avoided and has yet to be answered.

In Ref. [6] it is shown that, in the perfect concealing case ($\rho_B^{(0)} = \rho_B^{(1)}$), Alice can cheat and succeed for sure without knowing Bob's secret choices. In this paper, we present a general proof that unconditionally secure QBC is impossible even if Bob is allowed to generate probabilities unknown to Alice. Specifically we shall prove that, for any perfect or near-perfect concealing QBC protocol involving a secret probability distribution ω unknown to Alice, there exists a cheating unitary transformation independent of ω with which Alice can cheat.

Consider first the near-perfect case. Suppose we are given a protocol which is proven to be near-perfect concealing for whatever secret ω Bob chooses to use. Let

$$\omega = \{q_1, \dots, q_m\}, \quad (12)$$

where $q_j \geq 0$ and

$$\sum_{j=1}^m q_j = 1; \quad (13)$$

otherwise the q_j 's are arbitrary and unknown to Alice. Let Ω^* be a special set of distributions:

$$\Omega^* = \{\omega_1^*, \dots, \omega_m^*\}, \quad (14)$$

where

$$\omega_j^* = \{0, \dots, q_j = 1, \dots, 0\}. \quad (15)$$

The near-perfect concealing property implies that

$$F(\rho_B^{(1)}(\omega_j^*), \rho_B^{(0)}(\omega_j^*)) = 1 - \delta_j^*, \quad (16)$$

where $\delta_j^* > 0$, and $\delta_j^* \rightarrow 0$ asymptotically as the security parameter $N \rightarrow 0$ for all ω_j^* in Ω^* . It then follows from previous arguments that, for each ω_j^* , there exists a cheating unitary transformation $U_A(\omega_j^*)$, such that

$$|\langle \Psi_{AB}^{(1)}(\omega_j^*) | U_A(\omega_j^*) | \Psi_{AB}^{(0)}(\omega_j^*) \rangle| = 1 - \delta_j^*, \quad (17)$$

where $U_A(\omega_j^*)$ depends on ω_j^* in general.

Since ω is not revealed to Alice, Bob can purify his options with an arbitrary probability distribution over any set of possible choices. Consider the following purification over Ω^* ,

$$|\Psi_{AB}'^{(b)}\rangle = \sqrt{1/m} \sum_{j=1}^m |\Psi_{AB}^{(b)}(\omega_j^*)\rangle |\xi_j\rangle, \quad (18)$$

where $\{|\xi_j\rangle\}$ is a set of orthonormal ancilla states. The corresponding reduced density matrix,

$$\rho_B'^{(b)} = \text{Tr}_A |\Psi_{AB}'^{(b)}\rangle \langle \Psi_{AB}'^{(b)}|, \quad (19)$$

should also satisfy the near-perfect concealing condition

$$F(\rho_B'^{(1)}, \rho_B'^{(0)}) = 1 - \delta', \quad (20)$$

where $\delta' > 0$, and $\delta' \rightarrow 0$ as $N \rightarrow \infty$. Hence, as explained before, there exists a cheating unitary transformation U'_A , such that

$$\langle \Psi_{AB}'^{(1)} | U'_A | \Psi_{AB}'^{(0)} \rangle = 1 - \delta', \quad (21)$$

where the phase factor has been absorbed into U'_A for convenience. Notice that U'_A is independent of any secret parameters, so it is known to Alice. We shall show that Alice can use this U'_A to cheat, no matter how Bob purifies his secret choice of ω .

Substituting Eq. (18) into Eq. (21), we get

$$\frac{1}{m} \sum_{j=1}^m \langle \Psi_{AB}^{(1)}(\omega_j^*) | U'_A | \Psi_{AB}^{(0)}(\omega_j^*) \rangle = 1 - \delta'. \quad (22)$$

Let

$$\langle \Psi_{AB}^{(1)}(\omega_j^*) | U'_A | \Psi_{AB}^{(0)}(\omega_j^*) \rangle = (1 - \alpha_j) + i\beta_j, \quad (23)$$

where α_j and β_j are real, and $\alpha_j > 0$; then one can show that $\delta' \rightarrow 0$ if and only if every $\alpha_j \rightarrow 0$ and $\beta_j \rightarrow 0$. Intuitively this must be true because the two vectors, $|\Psi_{AB}'^{(1)}\rangle$ and $U'_A |\Psi_{AB}'^{(0)}\rangle$, can be nearly identical if and only if the corresponding orthogonal components, $|\Psi_{AB}^{(1)}(\omega_j^*)\rangle$ and $U'_A |\Psi_{AB}^{(0)}(\omega_j^*)\rangle$, are all nearly identical. This statement can be made quantitative as follows. Substituting Eqs. (23) into Eq. (22), we get

$$\delta' = \frac{1}{m} \sum_{j=1}^m \alpha_j, \quad (24)$$

and

$$\sum_{j=1}^m \beta_j = 0. \quad (25)$$

Eq. (24) shows that $\delta' \rightarrow 0$ if and only if all $\alpha_j \rightarrow 0$ as $N \rightarrow \infty$; furthermore each α_j should approach zero at least as fast as δ' . Hence α_j must satisfy

$$\alpha_j \leq c\delta', \quad (26)$$

where $0 < c \leq m$ is a constant independent of N . The fact that

$$|\langle \Psi_{AB}^{(1)}(\omega_j^*) | U'_A | \Psi_{AB}^{(0)}(\omega_j^*) \rangle| < 1 \quad (27)$$

implies

$$(\alpha_j^2 + \beta_j^2)/2 < \alpha_j \leq c\delta', \quad (28)$$

hence $\beta_j \rightarrow 0$ as $\alpha_j \rightarrow 0$. Then we have

$$\begin{aligned} |\langle \Psi_{AB}^{(1)}(\omega_j^*) | U'_A | \Psi_{AB}^{(0)}(\omega_j^*) \rangle|^2 &= 1 - 2\alpha_j + \alpha_j^2 + \beta_j^2, \\ &> 1 - 2c\delta', \end{aligned} \quad (29)$$

This result shows that, for any ω_j^* in Ω^* , Alice can use U'_A to cheat and her success probability is arbitrarily close to unity. That means, for practical purpose, Alice can use U'_A in place of the optimal but unknown $U_A(\omega_j^*)$ in Eq. (17), even though the two transformations may not be exactly equal.

Next we show that Alice can use U'_A to cheat even if Bob uses an arbitrary ω as given in Eq. (12). By definition, $|\Psi_{AB}^{(b)}(\omega)\rangle$ is a purification over the set Ω^* [see Eq. (14)], *viz.*,

$$|\Psi_{AB}^{(b)}(\omega)\rangle = \sum_{j=1}^m \sqrt{q_j} |\Psi_{AB}^{(b)}(\omega_j^*)\rangle |\xi_j\rangle. \quad (30)$$

Therefore according to Eq. (23),

$$\begin{aligned} \langle \Psi_{AB}^{(1)}(\omega) | U'_A | \Psi_{AB}^{(0)}(\omega) \rangle &= \sum_{j=1}^m q_j \langle \Psi_{AB}^{(1)}(\omega_j^*) | U'_A | \Psi_{AB}^{(0)}(\omega_j^*) \rangle, \\ &= 1 - \bar{\alpha} + i\bar{\beta}, \end{aligned} \quad (31)$$

where

$$\bar{\alpha} = \sum_{j=1}^m q_j \alpha_j, \quad (32)$$

$$\bar{\beta} = \sum_{j=1}^m q_j \beta_j. \quad (33)$$

From Eq. (26) and Eq. (32), we get

$$\bar{\alpha} \leq c\delta', \quad (34)$$

which, together with

$$|\langle \Psi_{AB}^{(1)}(\omega) | U'_A | \Psi_{AB}^{(0)}(\omega) \rangle| < 1, \quad (35)$$

gives

$$(\bar{\alpha}^2 + \bar{\beta}^2)/2 < \bar{\alpha} \leq c\delta'. \quad (36)$$

Then

$$\begin{aligned} |\langle \Psi_{AB}^{(1)}(\omega) | U'_A | \Psi_{AB}^{(0)}(\omega) \rangle|^2 &= 1 - 2\bar{\alpha} + \bar{\alpha}^2 + \bar{\beta}^2, \\ &> 1 - 2c\delta'. \end{aligned} \quad (37)$$

Consequently Alice can use U'_A to cheat, independent of what ω Bob chooses to use. We emphasize that U'_A may not necessarily maximize the quantity $|\langle \Psi_{AB}^{(1)}(\omega) | U'_A | \Psi_{AB}^{(0)}(\omega) \rangle|$, nevertheless Eq. (37) shows that Bob can use it to achieve the cheating purpose for arbitrary ω .

Finally we show that this same U'_A also works if Bob purifies his choices over an arbitrary set of ω 's, $\Omega = \{\omega_1, \dots, \omega_n\}$, where

$$\omega_k = \{q_1^k, \dots, q_m^k\} \quad (38)$$

as shown in Eq. (12). A general purification over Ω can be written as

$$|\Psi_{AB}''^{(b)}\rangle = \sum_{k=1}^n \sqrt{p_k} |\Psi_{AB}^{(b)}(\omega_k)\rangle |\chi_k\rangle, \quad (39)$$

where $|\Psi_{AB}^{(b)}(\omega_k)\rangle$ is given by Eq. (30), $|\chi_k\rangle$'s are orthonormal ancilla states, and $\{p_1, \dots, p_n\}$ is any probability distribution such that

$$\sum_{k=1}^n p_k = 1. \quad (40)$$

Then following the arguments presented earlier, we get

$$|\langle \Psi_{AB}''^{(1)} | U'_A | \Psi_{AB}''^{(0)} \rangle|^2 > 1 - 2c\delta'. \quad (41)$$

This result can also be easily obtained as follows. By a redefinition of the ancilla states, we can rewrite $|\Psi_{AB}''^{(b)}\rangle$ in terms of a single effective distribution [6]:

$$|\Psi_{AB}''^{(b)}\rangle = |\Psi_{AB}^{(b)}(\omega'')\rangle, \quad (42)$$

where $\omega'' = \{q_1'', \dots, q_m''\}$ is given by

$$q_j'' = \sum_{k=1}^n p_k q_j^k. \quad (43)$$

Then Eq. (41) follows directly from Eq. (37). Thus we conclude that, for any near-perfect concealing QBC protocol, Alice can use U'_A of Eq. (21) as the cheating transformation, no matter how Bob purifies his secret choices. In all cases, she succeeds with a probability $P_A(N)$ that can be made arbitrarily close to one by increasing the security parameter N .

It is straightforward to extend the above proof to cover the perfect concealing case as well. The perfect concealing condition, Eq. (3), implies that

$$\delta_j^* = 0 \quad (44)$$

in Eq. (17), and

$$\delta' = 0 \quad (45)$$

in Eq. (21). It then follows from Eq. (22) that

$$\langle \Psi_{AB}^{(1)}(\omega_j^*) | U'_A | \Psi_{AB}^{(0)}(\omega_j^*) \rangle = 1 \quad (46)$$

for all $\omega_j^* \in \Omega^*$. Hence

$$\alpha_j = \beta_j = 0 \quad (47)$$

in Eq. (23), and

$$\bar{\alpha} = \bar{\beta} = 0 \quad (48)$$

in Eq. (31). The above results imply that

$$U'_A = U_A(\omega) \quad (49)$$

for arbitrary ω , and the success probability $P_A(N) = 1$. Therefore if $\rho_B^{(0)} = \rho_B^{(1)}$, then Alice can use U'_A to cheat and succeed with probability equal to one, independent of Bob's secret choices.

Finally we note that the question of whether U'_A depends on Bob's ancilla states has also been raised [7]. The fact that it does not can be seen as follows. We know that any two different sets of ancilla states on Bob side are related by a unitary transformation U_B acting on Bob's Hilbert space H_B . Since

$$[U_B, U'_A] = 0, \quad (50)$$

it is obvious that U'_A does not depend on the particular ancilla set Bob chooses to use.

IV. CONCLUSION

In this paper we have proved that, for any perfect or near-perfect concealing QBC protocol involving a probability distribution ω unknown to Alice, there exists an ω -independent unitary transformation with which Alice can cheat. Our result closes a gap in the original impossibility proof [2, 3, 4, 5]. We conclude that, for those protocols covered by the original proof, unconditionally secure QBC is impossible even if Bob employs secret parameters.

APPENDIX

Suppose a protocol specifies that Bob should take certain action V_j ($j = 1, \dots, m$) on a state $|\phi\rangle$, according to a probability distribution $\omega_0 = \{q_1^0, \dots, q_m^0\}$. In the purified form, the resultant state is given by

$$|\psi(\omega_0)\rangle = \sum_{j=1}^m \sqrt{q_j^0} |\xi_j\rangle V_j |\phi\rangle, \quad (51)$$

where $|\xi_j\rangle$'s are orthonormal ancilla states. As shown in Ref. [6], a superposition of $|\psi(\omega_k)\rangle$'s, where $\omega_k = \{q_1^k, \dots, q_m^k\}$, can effectively be written in terms of a single distribution, *i.e.*,

$$|\psi'\rangle = \sum_{k=1}^n \sqrt{p_k} |\chi_k\rangle |\psi(\omega_k)\rangle \quad (52)$$

$$= |\psi(\omega')\rangle, \quad (53)$$

where $|\chi_k\rangle$'s are ancilla states, $\{p_1, \dots, p_n\}$ is a probability distribution, and $\omega' = \{q'_1, \dots, q'_m\}$ is the effective distribution given by

$$q'_j = \sum_{k=1}^n p_k q_j^k. \quad (54)$$

Let $\omega' = \omega_0$, then it is clear that Bob could generate $|\psi'\rangle$ instead of $|\psi(\omega_0)\rangle$, and he would have no problem passing any possible checks initiated by Alice. In general some qubits are measured and discarded in the checking procedure. For the remaining qubits, Bob could either stay with ω_0 , or he could collapse the ancillas $\{|\chi_k\rangle\}$ in Eq. (52) to obtain a state $|\psi(\omega_i)\rangle$, where ω_i is not equal to ω_0 in general.

Hence it is not meaningful for Alice to specify a probability distribution to an untrustful Bob, because there is no way to enforce it.

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